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An action approach to nodal and least energy normalized solutions for NLS Seminario de Análisis y Geometría Pontifical Catholic University of Chile

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Joint work with Colette De Coster (CERAMATHS/DMATHS, Valenciennes, France), Simone Dovetta and Enrico Serra (Politecnico di Torino, Italy)

Thursday 8 May 2025

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you!

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The nonlinear Schrödinger evolution equation

We consider the problem

$$\begin{cases} i\partial_t \psi = -\Delta \psi - |\psi|^{p-2}\psi, & (t,x) \in [0, T[\times \Omega, \\ \psi(t,x) = 0, & (t,x) \in [0, T[\times \partial \Omega, \\ \psi(0,x) = \psi_0(x), & \psi_0 : \rightarrow \mathbb{C}, x \in \Omega \end{cases}$$
(NLS_{evol})

where

$$ψ : [0, T[× Ω → ℂ, Ω bounded domain in ℝN, N ≥ 1;$$

$$i2 = -1;$$

• $\partial_t \psi$ is the derivative with respect to the time variable;

•
$$\Delta = \sum_{1 \le i \le N} \partial_{x_i}^2$$
 is the Laplacian on Ω ;

• p > 2 is a real parameter.

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Conservation laws

At least formally, the L^2 norm (the mass)

$$\|\psi(t,\cdot)\|_{L^2}^2 := \int_{\Omega} |\psi(t,x)|^2 \,\mathrm{d}x$$

and the energy

$$E\Big(\psi(t,\cdot)\Big) := \frac{1}{2} \int_{\Omega} |\nabla_x \psi(t,x)|^2 \, \mathrm{d}x - \frac{1}{p} \int_{\Omega} |\psi(t,x)|^p \, \mathrm{d}x$$

are preserved during the evolution.

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Solitary wave solutions

Opposed to blow-up: solitary waves of the form

$$\psi(t,x) = \mathrm{e}^{i\lambda t} u(x)$$

where $u \in H^1(\mathbb{R}^N; \mathbb{R}) = H^1(\mathbb{R}^N)$ is a solution of

$$-\Delta u + \lambda u = |u|^{p-2}u.$$
 (NLS)

Some vocabulary:

• $\lambda \in \mathbb{R}$ is the *frequency* of the solitary wave;

• $||u||_{L^2}^2 = ||\psi(t, \cdot)||_{L^2}^2$ is its mass.

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Two problems

Problem

Given $\lambda \in \mathbb{R}$, how to find a nonzero stationary wave of frequency λ ?

Problem

Given $\mu > 0$, how to find a stationary wave of mass μ ?

Vocabulary: solutions with a prescribed mass are usually called *normalized solutions*.

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Two functionals

We recall that the energy functional is given by

$$E(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, \mathrm{d}x - \frac{1}{p} \int_{\Omega} |u|^p \, \mathrm{d}x.$$

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Given $\lambda \in \mathbb{R}$, we also define the *action functional* by

$$\begin{aligned} J_{\lambda}(u) &:= E(u) + \frac{\lambda}{2} \int_{\Omega} |u|^2 \, \mathrm{d}x \\ &= \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, \mathrm{d}x + \frac{\lambda}{2} \int_{\Omega} |u|^2 \, \mathrm{d}x - \frac{1}{p} \int_{\Omega} |u|^p \, \mathrm{d}x. \end{aligned}$$

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Variational formulations

Proposition

Given $2 and <math>\lambda \in \mathbb{R}$, solutions of frequency λ correspond to critical points of J_{λ} on $H_0^1(\Omega)$.

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Proposition

Given $2 and <math>\mu > 0$, normalized solutions of mass μ correspond to constrained critical points of E on the L²-sphere

$$\mathcal{M}_{\mu} := \Big\{ u \in \mathcal{H}^1_0(\Omega) \mid \|u\|_{L^2(\Omega)} = \mu \Big\}.$$

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In the case of normalized solutions, the parameter λ in the PDE will appear as a Lagrange multiplier associated with the constraint.

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Lower boundedness of the energy functional

Proposition

Let $2 and <math>\mu > 0$. Then: if 2 , $inf <math>E > -\infty$; if 2 + 4/N , $inf <math>E = -\infty$.

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The boundedness follows from the Gagliardo-Nirenberg inequality

$$\|u\|_{L^p} \leq C(p) \|u\|_{L^2}^{1-s} \|\nabla u\|_{L^2}^s, \quad s := \frac{(p-2)N}{2p}.$$

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and the unboundedness by considering the limit $t \to +\infty$ for a family $t^{N/2}\psi(tx)$, with constant L^2 -norms, obtained by scaling a fixed profile.

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Proposition

When $\mu > 0$ and $2 , then minimizers for E on <math>\mathcal{M}_{\mu}$ exist, have a constant sign and are normalized solutions of (NLS). They are called energy ground states.

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Question

Given $\mu > 0$ and $2 + 4/N , do there exist normalized solutions of mass <math>\mu$? Is there a least energy normalized solution?

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How to find sign-changing normalized solutions?

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How to find sign-changing normalized solutions?

Answers: given by the results of the talk!

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The fixed frequency case

In the fixed frequency case, we are a priori looking for critical points of an unconstrained functional.

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The fixed frequency case

In the fixed frequency case, we are a priori looking for critical points of an unconstrained functional.

However, the functional J_{λ} is not bounded from below on $H_0^1(\Omega)$, since if $u \neq 0$ then

$$J_{\lambda}(tu) = \frac{t^2}{2} \|\nabla u\|_{L^2(\Omega)}^2 + \frac{\lambda t^2}{2} \|u\|_{L^2(\Omega)}^2 - \frac{t^p}{p} \|u\|_{L^p(\Omega)}^p \xrightarrow[t \to +\infty]{} -\infty.$$

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The Nehari manifold

A common strategy is to introduce the Nehari manifold \mathcal{N}_{λ} , defined by

$$\begin{split} \mathcal{N}_{\lambda} &:= \Big\{ u \in H_0^1(\Omega) \setminus \{0\} \mid J_{\lambda}'(u)[u] = 0 \Big\} \\ &= \Big\{ u \in H_0^1(\Omega) \setminus \{0\} \mid \|\nabla u\|_{L^2(\Omega)}^2 + \lambda \|u\|_{L^2(\Omega)}^2 = \|u\|_{L^p(\Omega)}^p \Big\}. \end{split}$$

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If $u \in \mathcal{N}_{\lambda}$, then

$$J_{\lambda}(u) = \Big(rac{1}{2} - rac{1}{p}\Big) \|u\|_{L^p(\Omega)}^p.$$

In particular, J_{λ} is bounded from below on \mathcal{N}_{λ} .

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Proposition

Given $\lambda > -\lambda_1(\Omega)$ and $2 , then minimizers for <math>J_{\lambda}$ on \mathcal{N}_{λ} exist, have a constant sign and are solutions of (NLS) having frequency λ . They are called action ground states.

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One defines the nodal Nehari set by

$$\mathcal{N}_{\lambda}^{nod} := \Big\{ u \in H_0^1(\Omega) \mid u^{\pm} \in \mathcal{N}_{\lambda}(\Omega) \Big\}.$$

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It contains all sign-changing solutions of (NLS).

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Theorem (Castro, Cossio, Neuberger 1997; Bartsch-Weth 2003)

Given $\lambda > -\lambda_2(\Omega)$ and $2 , then minimizers for <math>J_{\lambda}$ on $\mathcal{N}_{\lambda}^{nod}$ exist, have two nodal zones and are solutions of (NLS) having frequency λ . They are called nodal action ground states.

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Remark: I will use the terms "sign-changing" and "nodal" interchangeably, as the contrary of "one-signed".

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Comparison of the two settings so far

Abbreviation: "ground state" \rightarrow GS

	2	$2 + 4/N$
Positive solution	Energy GS	?
Sign-changing solution	?	?

The fixed mass μ case

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	2	$2 + 4/N$
Positive solution	Action GS	Action GS
Sign-changing solution	Nodal action GS	Nodal action GS

The fixed action λ case

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Sign-changing normalized solutions

The only work I am aware of which studies sign-changing normalized solutions is a recent preprint of Jeanjean and Song in 2025, using gradient flow techniques.

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Sign-changing normalized solutions

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In the literature, there is no equivalent of the nodal Nehari set for normalized solutions and it is in fact very unclear if such a nice "codimension two constraint" does exist for this problem.
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Two main difficulties have to be taken into account:

 \blacksquare the energy is unbounded from below on the constraint \rightarrow one obtains "mountain pass" solutions;

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To resolve the second issue, one either uses the Pohožaev identity or a monotonicty trick "à la Struwe", later improved by Jeanjean and coauthors.

While remarkably successful for autonomous PDEs set on \mathbb{R}^N , those techniques impose a lot of restrictions on the domain under study.

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Noris-Tavares-Verzini 2014 studied this problem on the ball, relying heavily on the uniqueness of the positive solution for a given λ (based on seminal results of Gidas-Ni-Nirenberg and Kwong). Their analysis is very precise, but limited to a specific domain;

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- Pierotti-Verzini 2016 then Pierotti-Verzini-Yu 2025 considered general bounded domains. Good existence results require star-shapedness of the domain, a quite strong geometrical assumption.

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Notably, the authors point out that, in the L^2 -supercritical regime on a bounded domain, sequences of solutions having a bounded Morse index are bounded in L^2 .

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Action versus energy ground states

While both notions of action GS and of energy GS have a long history, most papers studied either one or the other, and a comparison of both notions was not considered until rather recently.

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Proposition (Dovetta-Serra-Tilli^(*) 2022)

Let $2 and <math>\Omega$ be bounded.

Then if energy ground states do exist, they are necessarily action ground states for the corresponding λ . The converse is not necessarily true!

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Then if energy ground states do exist, they are necessarily action ground states for the corresponding λ . The converse is not necessarily true!

(*) This statement was more or less known in the literature before the DST paper, but not considered from the point of view of the systematic comparison of both notions of GS.

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Action versus energy ground states (continued)

Theorem (Dovetta-Serra-Tilli 2022)

Let $2 and <math>\Omega$ be bounded.

For any $\mu > 0$, define

$$\mathcal{E}(\mu) := \inf_{u \in \mathcal{M}_{\mu}} E(u)$$

and, for every $\lambda \in \mathbb{R}$, define

$$\mathcal{J}(\lambda) := \inf_{u \in \mathcal{N}_{\lambda}} J_{\lambda}(u).$$

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Then, $-\mathcal{E}(2\mu)$ is the Legendre-Fenchel transform of \mathcal{J} . Namely, one has

$$-\mathcal{E}(2\mu) = \sup_{\lambda \in \mathbb{R}} (\lambda \mu - \mathcal{J}(\lambda)).$$

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In their paper, Dovetta, Serra and Tilli *compare two families of solutions* whose existence is known a priori via minimization procedures: the action GS and the energy GS.

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Main message

The convex duality we just saw is a method !!!

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The convex duality we just saw is a method !!!

More precisely:

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More precisely:

- using such a "convex duality argument" from the action ground states when 2 + 4/N will*also*produce normalized solutions;
- doing so from the nodal action GS will produce sign-changing normalized solutions, which is new for all 2

Our result (for positive solutions)

Theorem (De Coster-Dovetta-G.-Serra 2025)

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and, for every 2 , let

$$M_{p} := \left\{ \|u\|_{L^{2}(\Omega)}^{2} \mid u \in \mathcal{N}_{\lambda}(\Omega) \text{ and } J_{\lambda}(u) = \mathcal{J}(\lambda) \text{ for some } \lambda \in \mathbb{R}
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be the set of masses of all action ground states. Then,

- (i) if $2 , then <math>M_p(\Omega) = (0, +\infty)$;
- (ii) if $2 + 4/N , then there exist <math>0 < \mu_p < +\infty$ such that $M_p = (0, \mu_p]$.

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Isn't that quite obvious?

One may argue that obtaining *intervals of masses* is a trivial consequence of the intermediate value theorem.

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One may argue that obtaining *intervals of masses* is a trivial consequence of the intermediate value theorem.

This would be true *if* the map $\lambda \mapsto u_{\lambda}$ mapping λ to the action GS had good continuity properties, *which is expected to be wrong in general!*

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One may argue that obtaining *intervals of masses* is a trivial consequence of the intermediate value theorem.

This would be true *if* the map $\lambda \mapsto u_{\lambda}$ mapping λ to the action GS had good continuity properties, *which is expected to be wrong in general!*

In fact, this map is not even well-defined as action GS might not be unique.

Foreword	NLS	State of the art	The "action approach"	Least energy normalized solutions

Proposition

Let 2 . Then:

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(i) For every $\lambda \leq -\lambda_1$, $\mathcal{J}(\lambda) = 0$ and action ground states in $\mathcal{N}_{\lambda}(\Omega)$ do not exist.

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- (i) For every $\lambda \leq -\lambda_1$, $\mathcal{J}(\lambda) = 0$ and action ground states in $\mathcal{N}_{\lambda}(\Omega)$ do not exist.
- (ii) For every $\lambda > -\lambda_1$, $\mathcal{J}(\lambda) > 0$ and action ground states in $\mathcal{N}_{\lambda}(\Omega)$ exist.

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- (iii) The function $\mathcal{J} : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and increasing on $[-\lambda_1, +\infty)$.

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- (iii) The function $\mathcal{J} : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and increasing on $[-\lambda_1, +\infty)$.

Moreover, "derivatives of \mathcal{J} give L^2 -masses of action ground states" (to be precised).

Foreword	NLS	State of the art	The "action approach"	Least e

_east energy normalized solutions

A completely wrong argument But still a good heuristic :-)

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Foreword	NLS	State of the art	The "action approach"	Leas

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Recalling the definition of J_{λ} , we have

$$J_{\lambda}(u) = E(u) + \frac{\lambda}{2} \|u\|_{L^2}^2,$$

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Of course, we have that $\mathcal{J}(\lambda) = J_{\lambda}(u_{\lambda})$ for a **varying** action GS u_{λ} (they must be in different Nehari manifolds!). It just so happens that the action GS change "little enough" that the leading term is the same than if the minimizer was fixed, which is extremely convenient.

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A correct version of the heuristic argument

Proposition

Let 2 and define

$$Q_p(\lambda) := \Big\{ \|u\|_2^2 \mid u \in \mathcal{N}_\lambda(\Omega) \text{ and } J_\lambda(u,\Omega) = \mathcal{J}(\lambda) \Big\}.$$

be the set of masses of action ground states.

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$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{J}(\lambda + \varepsilon) - \mathcal{J}(\lambda)}{\varepsilon} = \frac{1}{2} \inf Q_p(\lambda)$$
$$\leq \frac{1}{2} \sup Q_p(\lambda) = \lim_{\varepsilon \to 0^-} \frac{\mathcal{J}(\lambda + \varepsilon) - \mathcal{J}(\lambda)}{\varepsilon},$$

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Moreover, for every λ outside an at most countable set, all action ground states have the same mass (i.e., $Q_p(\lambda)$ is a singleton).
Foreword	NLS	State of the art	The "action approach"	Least energy normalized solutions
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A miracle

Proposition (Key proposition)

Let $\mu > 0$ and $2 . Assume that <math>\lambda_* > -\lambda_1(\Omega)$ is a local minima of the map $f_{\mu} : [-\lambda_1, +\infty) \to \mathbb{R}$ defined by

$$f_{\mu}(\lambda) := \mathcal{J}(\lambda) - rac{1}{2}\mu\lambda.$$

Then, \mathcal{J} is differentiable for $\lambda = \lambda_*$ and one has that $\mathcal{J}'(\lambda_*) = \mu$, so that all action ground states with $\lambda = \lambda_*$ have mass μ .

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Proof of the key proposition

Proof.

At a minimum point, one must have

$$\limsup_{\varepsilon \to 0^{-}} \frac{f_{\mu}(\lambda_{*} + \varepsilon) - f_{\mu}(\lambda_{*})}{\varepsilon} \leq 0 \leq \liminf_{\varepsilon \to 0^{+}} \frac{f_{\mu}(\lambda_{*} + \varepsilon) - f_{\mu}(\lambda_{*})}{\varepsilon},$$

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But we just saw that the reverse inequality holds!

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Interlude: Darboux's Theorem for derivatives

Somehow, we just proved a "Darboux-type" result theorem for \mathcal{J}' (even though \mathcal{J}' is not pointwise well-defined). As a comparison, here is Darboux's original theorem.

Theorem (Darboux 1875)

Let $f : I \to \mathbb{R}$ be differentiable, where I is an interval. Then, f'(I) is an interval.

Foreword	NLS	State of the art	The '
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The "action approach"

Least energy normalized solutions

Jean-Gaston Darboux (1842 – 1917)



Image from Wikimedia Commons.

Foreword	NLS	State of the art	The "action approach"	Least energy normalized solutions

Asymptotic behavior of $\mathcal{J}: \lambda \to -\lambda_1$

Proposition

For every $2 , there exist <math>C_1, C_2 > 0$ such that for every $\lambda \ge -\lambda_1$,

$$egin{split} \mathcal{J}(\lambda) &\leq C_1 (\lambda+\lambda_1)^{rac{p}{p-2}} \ \mathcal{J}(\lambda) &\geq C_2 \min\left(1,rac{\lambda+\lambda_1}{\lambda_1}
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In particular,

$$\frac{\mathcal{J}(\lambda)}{\lambda+\lambda_1} \xrightarrow[\lambda \to -\lambda_1]{} 0.$$

Foreword	NLS	State of the art	The "action approach"	Least energy normalized solution

Asymptotic behavior of $\mathcal{J}: \lambda \to +\infty$

Proposition

We have

$$\lim_{\lambda \to +\infty} \frac{\mathcal{J}(\lambda)}{\lambda} = \begin{cases} +\infty & \text{if } 2$$

Foreword	NLS	State of the art	The "action approach"	Least energy normalized solutions

Using the asymptotic results, we are able to show that the map $\lambda \mapsto \mathcal{J}(\lambda) - \frac{1}{2}\mu\lambda$ has local minima:

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Using the asymptotic results, we are able to show that the map $\lambda \mapsto \mathcal{J}(\lambda) - \frac{1}{2}\mu\lambda$ has local minima:

■ for all 0 < µ if 2 < p < 2 + 4/N, in this case one can even find global minima;</p>

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This proves our announced results for positive solutions.

Foreword	NLS	State of the art	The "action approach"	Least energy normalized solut

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- life gets harder when $\lambda \leq -\lambda_1(\Omega)$, essentially because the quadratic form

$$u \mapsto \int_{\Omega} |\nabla u|^2 \, \mathrm{d}x + \lambda \int_{\Omega} |u|^2 \, \mathrm{d}x$$

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- we have to rely on Bartsch-Weth's (non-trivial!) result to obtain existence of nodal action ground states when $-\lambda_2 < \lambda \leq -\lambda_1$;
- the claims can be adapted quite naturally to the nodal setting and proved in analogous ways, up to the above remarks. I refer to the paper for details!

Foreword	NLS	State of the art	The "action approach"	Least energy normalized solutions

What are we looking for?

When 2 + 4/N , we saw that the energy functional is unbounded from below on the mass constraint.

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When 2 + 4/N , we saw that the energy functional is unbounded from below on the mass constraint.

We may however be interested in least energy normalized (nodal) solutions, namely solutions having least energy among all (nodal) solutions.

For instance, Jeanjean's seminal 1997 paper produces least energy normalized solutions on \mathbb{R}^N .

Foreword	NLS	State of the art	The "action approach"	Least energy normalized solutions

Pohožaev's identity

The following identity is often useful in the study of semilinear elliptic PDEs and follows by *clever* integration by parts.

Proposition (Pohožaev's identity, 1965)

Let 2 have a smooth boundary and <math display="inline">u be a solution to (NLS). Then, one has

$$\frac{N-2}{2} \|\nabla u\|_{2}^{2} - \frac{N}{p} \|u\|_{p}^{p} + \frac{\lambda N}{2} \|u\|_{2}^{2} + \frac{1}{2} \int_{\partial \Omega} |\partial_{\nu} u|^{2} x \cdot \nu \, d\sigma = 0.$$

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Remark: when $\Omega = \mathbb{R}^N$, there is no boundary term! This is why this identity is much more powerful on \mathbb{R}^N than on domains.

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Star-shaped domains

Corollary

If Ω is star-shaped, then

$$\frac{N-2}{2} \|\nabla u\|_2^2 - \frac{N}{p} \|u\|_p^p + \frac{\lambda N}{2} \|u\|_2^2 \le 0.$$

Corollary

If Ω is star-shaped and u is a solution of (NLS), then

$$E(u) \geq \frac{N(p-p_c)}{4p} ||u||_p^p, \quad p_c := 2 + 4/N.$$

In particular, on star-shaped domains, all solutions have a positive energy in the L^2 -supercritical case!

Damien Galant

The result (for positive solutions)

Theorem (De Coster-Dovetta-G.-Serra 2025)

Let Ω be bounded, open, smooth and star-shaped and 2 . Then:

- if 2 < p < 2 + 4/N, then least energy normalized (nodal) solutions do exist for all masses;
- if 2 + 4/N < p < 2*, then least energy normalized (nodal) solutions do exist for all small masses.

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Main idea: using the consequences of Pohožaev's identity, we show that solutions having a small mass must correspond to λ close enough to $-\lambda_1$ (for GS) or to $-\lambda_2$ (for nodal GS), corresponding to cases we can handle with the "action approach".

Foreword	NLS	State of the art	The "action approach"	Least energy normalized solutions

One can show that least energy solutions (resp. least energy nodal solutions) exist for all $\mu \in (0, \mu_N)$ (there are possibly more), resp. for all $\mu \in (0, 2\mu_N)$, where μ_N is the mass of the corresponding soliton on \mathbb{R}^N .

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Moreover, the analysis of Noris-Tavares-Verzini on the ball implies that the masses of all positive solutions are given *exactly* by $(0, \mu_N)$.

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Thus, on the ball, for $\mu \in [\mu_N, 2\mu_N)$, least energy nodal solutions exist, and there are no positive solutions, so that...

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In the critical and supercritical cases...

least energy solutions may exist and be nodal!

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In the critical and supercritical cases...

least energy solutions may exist and be nodal!

This strikingly shows that not all properties of energy ground states transfer to least energy normalized solutions.

A (difficult?) open question

If Ω is not star-shaped, it is known that negative energy solutions can exist. This can be explored by studying such problems on metric graphs, which often lead to "simple" non-star-shaped domains.

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Question

Is there an intricate smooth bounded domain Ω , an exponent $2 + 4/N and a mass <math>\mu$ for which there exist a sequence of normalized solutions of mass μ whose energy go to $-\infty$?

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Question

Is there an intricate smooth bounded domain Ω , an exponent $2 + 4/N and a mass <math>\mu$ for which there exist a sequence of normalized solutions of mass μ whose energy go to $-\infty$?

My guess ... maybe yes, actually?

¡Muchas gracias!

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